Equivalent spring constant

Series springs:

\[
\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}
\]

Parallel springs:

\[
k = k_1 + k_2
\]
Typical support beams in MEMS

1- Cantilever:

\[ k_z = \frac{3EI_z}{L^3} = \frac{Eh^3w}{4L^3} \]

\[ k_z = 2 \times \frac{3EI_z}{L^3} = \frac{Eh^3w}{2L^3} \]
Typical support beams in MEMS

2- Bridge:

\[ k_y = 2 \times \frac{12EI_y}{L^3} = \frac{2Ew^3h}{L^3} \]
\[ k_z = 2 \times \frac{12EI_z}{L^3} = \frac{2Eh^3w}{L^3} \]
Typical support beams in MEMS

3- Crab leg:

\[ k_y = \infty \]
\[ k_z = 4 \times \frac{12EI_z}{L^3} = \frac{4Eh^3w}{L^3} \]
4- Folded:

Shuttle flexure: \( k_s = k_3 + k_4 = 2 \times \frac{12EI}{L^3} \)

Anchored flexure:

\[ k_a = k_1 + k_2 = 2 \times \frac{12EI}{L^3} \]

Typical support beams in MEMS
Torsion

\[ \sum M = 0 \rightarrow T = \iint r\tau (rd\theta dr) = \iint rG\gamma (rd\theta dr) = \iint rG \frac{r\phi}{L} (rd\theta dr) = G \frac{\phi}{L} \iint r^2 (rd\theta dr) \]

\[ J = \iint r^2 dA \quad \text{Polar moment of inertia (second polar moment of area)} \]

\[ T = \frac{JG}{L} \phi \]
Residual stress

We know that when a material is deposited, it has some internal stress. When the microstructure is released, if there is no internal stress, the beam remains flat, but if there is internal stress, it can cause the microstructure to bend, as illustrated below.

![Before release](image1)

![After release](image2)

If there is no stress gradient in the cross section of a cantilever, the structure will not bend (The stress causes the beam to elongate or shrink); however, in many cases there is stress gradient:

\[
\sigma(z) = \sigma_1 \frac{z}{a/2}
\]

\[
M = \int_{-a/2}^{a/2} z\sigma(z)bdz = \sigma_1 \frac{b}{a/2} \int_{-a/2}^{a/2} z^2dz = \sigma_1 \frac{a^2b}{6}
\]

\[
\frac{1}{\rho} = \frac{M}{EI} = \frac{2\sigma_1}{E} \frac{1}{a} \quad \frac{d^2y}{dx^2} = \frac{2\sigma_1}{E} \frac{1}{a} \quad y = \frac{\sigma_1}{E} \frac{1}{a} x^2 + c_1x + c_0 \quad \rightarrow \quad y = \frac{\sigma_1}{E} \frac{1}{a} L^2
\]
Residual stress in bridges

In doubly-supported beams (i.e., bridges), residual stress causes the beam to either be in high tension if one has residual tension, or the beam to buckle and bend out of plane as shown if the residual stress is highly compressive.

For a beam, with fixed supports, the maximum stress before buckling occurs is given by:

$$\sigma_{\text{max}} = \frac{\pi^2 E a^2}{3 L^2}$$

Note that the type of support (fixed, or pinned) will significantly influence the maximum load before buckling, and so support conditions are very important.

Solutions for residual stress:

– Relieve the stress: the stress can be annealed out in many cases, or at least be reduced. By reducing the stress, it may be possible to avoid buckling and other stress related issues.

– In choosing device dimensions, and designing the mechanical structures, we have to account for the effects of stress, and so it is possible to make the structures strong enough to tolerate some level of stress, and therefore not buckle under the stress.

– Or, we can use structures that because of their design allow the stress to be relieved.